Pressure Dependent Yield Criteria Applied for Improving Design Practices and Integrity Assessments against Yielding of Engineering Polymers

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Manuscript received January 12, 2011 in revised form April 18, 2012

Conventional yield criteria for ductile materials, such as Tresca and von Mises, predict that yielding is independent on the hydrostatic stress state (pressure), which means that tensile and compressive stress-strain behaviors are considered equal and are equally treated. This approach is reasonable for ductile metallic materials but sometimes inaccurate for polymers, which commonly present larger compressive yield strength, therefore being characterized as uneven. Some pressure dependent theories are available, but there is no consensus concerning the choice of the most appropriate criterion, its use and benefits. As a step in the direction of improving structural integrity practices taking advantage of unevenness, this work performs three key-activities: i) first, a critical review about existing theories and its accuracy; ii) second, a series of experiments under tension and compression including four selected polymers (PA-66, PA-6, PP, and HDPE) to assess real unevenness levels; iii) third, a numerical evaluation of the potential benefits of using modified criteria. Stress states, safety, and stiffness were evaluated for a typical application to illustrate the proposals. Mass reductions up to 39% could be achieved even with simple geometric changes, while keeping original safety and stiffness levels.

KEY WORDS: Uneven polymers; Pressure dependent yield criteria; Experimental testing; Structural improvement

1. Introduction

Classical plasticity theories and yield criteria for ductile materials, such as Tresca and von Mises formulations\(^1\), include several assumptions, such as: i) material is isotropic and homogeneous; ii) deformation takes place under constant volume; iii) tensile and compressive yield strengths are equal; iv) yielding phenomenon is not influenced by the hydrostatic component of the stress state \((\sigma_0\text{ or pressure})^2\). The first two assumptions can be kept unchanged considering the interest of this work, while the others deserve a critical reflection. These last two assumptions imply that the behaviors of tensile and compressive stress-strain are identically treated in terms of structural integrity. While reasonable for ductile metallic materials, it can be inaccurate for polymers, ceramics, and even brittle metals. Engineering ductile thermoplastic polymers, focus of this study, usually present larger compressive yield strength, therefore being characterized as uneven polymers\(^2\). This is a direct result of chains arrangement and deformation micromechanisms, which are
dependent on the hydrostatic stress level\textsuperscript{[4,3]. For clarity, the unevenness level in terms of yield strength is denoted “m” and is defined here as:

\[ m = \frac{\sigma_{ys-t}}{\sigma_{ys-c}} \]  

where \( \sigma_{ys-t} \) and \( \sigma_{ys-c} \) represent the yield strength under tension and compression, respectively.

In spite of being very scarce until present days, some experimental results available in the literature, including tensile and compressive data, reveals that unevenness for experimental results available in the literature, including tensile and compression, respectively.

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where \( \sigma_{ys-t} \) and \( \sigma_{ys-c} \) represent the yield strength under tension and compression, respectively.

In spite of being very scarce until present days, some experimental results available in the literature, including tensile and compressive data, reveals that unevenness for polymers usually lies between m = 1.20 and m = 1.30\textsuperscript{[4,3]. Additional results by Jerabek et al.\textsuperscript{[9] show yield strength 50% larger under compression (m = 1.50) for polypropylene (PP). An exploratory investigation conducted by the authors using CES Edupack 2011 software database\textsuperscript{[10]} revealed that, for the available 298 unfilled thermoplastics, in most cases m varies from 1.00–1.80. For the materials tested in this study, the same database indicates m ranging from 0.90–1.60 for PA-66, 1.10–1.40 for PA-6, 1.10–1.45 for PP, and 0.95–1.50 for HDPE.

Unfortunately, these uneven mechanical properties are in general not considered by current design and integrity assessment practices, which are based on protocols developed during several decades for metallic materials. However, gains based on the simple substitution of metallic materials for polymers are becoming saturated and more laborious. In this context, a better understanding of the mechanical behavior of polymers supported by the use of pressure dependent yield criteria represents potential opportunities for structural improvement.

As a step in this direction, this work evaluates the effects of implementing pressure dependent yield criteria on design practices for components with regions working under compression. First, several polymers were tested under tension/compression to obtain real stress-strain curves and unevenness levels. In the sequence, analytical and numerical calculations including optimization procedures were developed to incorporate the different criteria in the design process and assess stiffness, weight, stresses and safety factors of an example component. Results show that pressure dependent criteria combined to the obtained data could reduce components’ mass up to ~39% while keeping original stiffness and safety factors against yielding.

2. Theoretical Framework - Pressure Dependent Yield Criteria

Several different pressure dependent yield criteria have been proposed, but most of them are based on the classic criterion proposed by Huber\textsuperscript{[11] and von Mises\textsuperscript{[12]} (Eq. (3)). Von Mises proposed that yielding occurs when the second invariant of the deviatoric stress tensor (\( J_2 \)) reaches a critical value (\( k^2 \))\textsuperscript{[13]}, in the form:

\[ J_2 = k^2; \quad J_2 = \frac{1}{6} (\sigma_{1} - \sigma_{2})^2 + (\sigma_{2} - \sigma_{3})^2 + (\sigma_{3} - \sigma_{1})^2, \quad k = \frac{1}{\sqrt{3}} \sigma_{ys} \]  

Yielding takes place if equivalent stress (\( \sigma_{eq} \)) is greater than tensile yield strength (\( \sigma_{ys} \)). The resulting yield locus for this criterion is presented by Fig. 1a, being \( \sigma_{1} \), \( \sigma_{2} \), and \( \sigma_{3} \) the three principal stresses. Since the hydrostatic stress can be written in terms of the first stress invariant (\( I_1 = \sigma_{1} + \sigma_{2} + \sigma_{3} \)) as \( \sigma_{h} = (I_1 / 3) \), it can be realized that there is no predicted effect of \( \sigma_{h} \) on this failure prediction (the locus of Eq. (3) is a cylindrical tube aligned with the hydrostatic axis - Fig. 1a).

To include the pressure dependency on von Mises original criterion, Hu and Pae\textsuperscript{[14]} included in Eq. (2) a second term depending on \( I_1 \), leading to Eq. (4). Expanding this formulation as a polynomial in \( I_1 \) and following Ehrenstein and Erhard\textsuperscript{[15]} and Miller\textsuperscript{[16]}, Eqs. (5) and (6) can be obtained for \( N = 1 \) and \( N = 2 \), respectively. Eq. (5) represents the conically modified von Mises (or Drucker-Prager) criterion, while Eq. (6) represents the parabolically modified von Mises criterion. In the conical model, Eq. (5) reveals that the \( I_1 \) effect is linear, providing the yield surface shown by Fig. 1b. In the parabolic model, in its turn, Eq. (6) reveals that the \( I_1 \) effect is quadratic, providing the yield surface of Fig. 1c. In both cases, the higher the compressive hydrostatic stress, the higher

\[ \sigma_{eq} = \frac{1}{\sqrt{2}} \sqrt{\left( \sigma_{1} - \sigma_{2} \right)^2 + \left( \sigma_{2} - \sigma_{3} \right)^2 + \left( \sigma_{3} - \sigma_{1} \right)^2} \]  

Fig. 1 Illustrative yield surfaces plotted relative to the three principal axes considering (a) classical von Mises; (b) conically modified von Mises; and (c) parabolically modified von Mises criteria.
is the predicted yield strength, and yielding occurs when the modified equivalent stress is equal to the tensile yield strength \( (\sigma_{\text{eq,C.P}} = \sigma_{y,s}) \). Additional improvement was proposed by Ghorbel in 2008\cite{17}, including the third invariant of the deviatoric stress tensor \( (J_2) \) in Eq. (4), but with slight benefits that will not be taken into account here.

\[
J_2 = k^2 + \sum_{n=1}^{\infty} \alpha_n \cdot I^n
\]  

\[
\sigma_{\text{eq,C.P}} = \frac{1}{2m} \left[ (m-1) \cdot I_1 + (m+1) \cdot \sqrt{\frac{3}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]} \right]
\]  

\[
\sigma_{\text{eq,C.P}} = \frac{m-1}{2m} \cdot I_1 + \frac{m-1}{2m} \cdot I_1^2 \cdot \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]
\]

Fig. 2a presents a comparison between the original and the two modified yield criteria for plane stress conditions and two \( m \) values. Conical model is clearly more sensitive to high \( m \) values, which is expected due to the linear dependence on \( I_1 \) and consequently \( \sigma_1 \). Both criteria are based on consistent approaches\cite{4–7,17}, however, the parabolic model is considered by many researchers as more realistic when compared to experimental results \cite{4–7,17} (Fig. 2b).

3. Tested Materials and Experimental Procedures

Four thermoplastic polymers were tested under tension and compression, including PA-66, PA-6, PP and HDPE. Samples from each material were obtained from the same bar (25.4 mm in diameter) to avoid different batches and were coincident to the bar centerline, in order to sample the same material characteristics. Machining was conducted using CNC with small passes (to avoid residual stresses or damage to the raw material) according to ASTM D638\cite{18} for tension (rectangular cross section with thickness = 7.0 mm and width = 13.0 mm) and ASTM D695\cite{19} for compression (cylinders with length = 25.4 mm and diameter = 12.7 mm).

The samples were kept and tested at 21ºC and 60% relative humidity, using the same strain rate for tensile and compressive testing (0.051 s\(^{-1}\)). Ten valid samples were tested for each material (being 5 tensile and 5 compressive). For all samples it was calculated the: i) elastic modulus (\( E \)); ii) offset yield strengths considering engineering (\( S_{\text{ys-offset}} \)) and true (\( S_{\text{ys-offtrue}} \)) stress-strain response for 0.2%, 0.5%, 1%, and 2% plastic strain offsets; iii) maximum engineering yield strength (\( S_{\text{ys-max}} \)) on the first point were \( dS/de = 0 \), being \( e \) the engineering strain.

4. Experimental Results and Discussion

Most tensile specimens (except from some PA-6 and PP) presented necking without fracture. Compressive specimens were monitored using real-time images and presented barreling and in some cases buckling, but only for strain levels far higher than necessary for this investigation. Stress-strain curves for all five samples tested, for each configuration and material, provided excellent agreement and are not presented here due to space limitations. Selected curves from representative samples of interest are presented by Fig. 3, which reveals that yield strength unevenness clearly exists for PA-6, PP and HDPE, as will be detailed and quantified next.

True stress (\( \sigma \)) versus true strain (\( \varepsilon \)) data could be calculated based on engineering results (\( S - \varepsilon \)) from Fig. 3 using Eq. (7)\cite{3,4}. Unevenness levels could consequently be calculated for both engineering (\( m_1 \)) and true (\( m_2 \)) data using Eq. (1)\cite{17,18,19}. However, true data reveals less unevenness and can be considered more realistic due to the large strain response of polymers even for low stress levels, being presented in details in Table 1. Fig. 4 summarizes average (\( m_1 \)) and (\( m_2 \)) values considering all offset levels. In spite of not being a physical measurement, these average values describe the unevenness behavior through elastic and the beginning of elastic-plastic loading and are representative of the real behavior of the material under tension and compression. Polypropylene, for example, presented 24% larger yield strength under compression while keeping stiffness. Calling Fig. 2a and considering this \( m = 1.24 \) for different \( \sigma: \sigma \) ratios (represented by \( \theta \)), Fig. 4b shows that failure prediction by yielding can be increased up to ~ 40%. Elastic modulus unevenness was also quantified in Table 1 as an adapted “\( m \)” and all materials presented Elastic Modulus reduction under compression.

\[
\sigma = S \cdot (1 + \varepsilon); \quad \varepsilon = \ln(1 + \varepsilon)
\]  

Fig. 2 (a) Yield loci for original von Mises criterion compared by the author to conically and parabolically modified models for \( m = 1.3 \) and \( m = 2.0 \). (b) Experiments compared to parabolic model prediction\cite{18}.
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Fig. 4 (a) Average values of \( m_e \) and \( m_t \) considering all offset levels from Table 1; (b) Example of potential gains for different \( \mu_1: \mu_2 \) ratios considering PP with \( m_t = 1.24 \)

<table>
<thead>
<tr>
<th>Tested Material</th>
<th>Unevenness - ( m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PA-66</td>
<td>1.05</td>
</tr>
<tr>
<td>PA-6</td>
<td>1.25</td>
</tr>
<tr>
<td>PP</td>
<td>1.36</td>
</tr>
<tr>
<td>HDPE</td>
<td>1.24</td>
</tr>
<tr>
<td>HDPE</td>
<td>1.21</td>
</tr>
<tr>
<td>HDPE</td>
<td>1.18</td>
</tr>
<tr>
<td>Reference</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Fig. 4 (a) Average values of \( m_e \) and \( m_t \) considering all offset levels from Table 1; (b) Example of potential gains for different \( \sigma_1: \sigma_2 \) ratios considering PP with \( m_t = 1.24 \)

Table 1 Results for tensile mechanical properties and unevenness levels, which were calculated here for true (\( m_t \)) stress-strain data. ND means that parameters are not defined by standards or formulations are not applicable. Compressive data can be calculated using Eq. (1)

<table>
<thead>
<tr>
<th>Material</th>
<th>E (MPa)</th>
<th>( \sigma_{p0.2} )</th>
<th>( \sigma_{p0.5} )</th>
<th>( \sigma_{p1.0} )</th>
<th>( \sigma_{p2.0} )</th>
<th>( \sigma_{p\text{max}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PA-66 Tensile</td>
<td>2766±270</td>
<td>48.3±7.1</td>
<td>59.8±4.1</td>
<td>65.9±1.7</td>
<td>67.0±1.0</td>
<td>67.2±1.1</td>
</tr>
<tr>
<td>( m_{t:PA-66} )</td>
<td>0.72±0.11</td>
<td>1.01±0.15</td>
<td>0.92±0.07</td>
<td>0.89±0.03</td>
<td>0.91±0.02</td>
<td>ND</td>
</tr>
<tr>
<td>PA-6 Tensile</td>
<td>2871±209</td>
<td>36.0±6.5</td>
<td>46.0±5.6</td>
<td>55.7±3.1</td>
<td>63.4±0.9</td>
<td>68.7±0.9</td>
</tr>
<tr>
<td>( m_{t:PA-6} )</td>
<td>0.82±0.08</td>
<td>1.16±0.18</td>
<td>1.08±0.12</td>
<td>1.06±0.06</td>
<td>1.09±0.02</td>
<td>ND</td>
</tr>
<tr>
<td>PP Tension</td>
<td>1778±112</td>
<td>20.7±0.6</td>
<td>25.0±0.4</td>
<td>28.6±0.5</td>
<td>33.0±0.8</td>
<td>39.2±0.3</td>
</tr>
<tr>
<td>( m_{t:PP} )</td>
<td>0.95±0.08</td>
<td>1.27±0.04</td>
<td>1.21±0.03</td>
<td>1.22±0.02</td>
<td>1.24±0.03</td>
<td>ND</td>
</tr>
<tr>
<td>HDPE Tension</td>
<td>1650±93</td>
<td>9.0±0.5</td>
<td>12.7±0.5</td>
<td>16.2±0.5</td>
<td>19.9±0.4</td>
<td>26.0±0.8</td>
</tr>
<tr>
<td>( m_{t:HDPE} )</td>
<td>0.56±0.11</td>
<td>1.42±0.10</td>
<td>1.19±0.06</td>
<td>1.08±0.04</td>
<td>1.02±0.03</td>
<td>ND</td>
</tr>
</tbody>
</table>
5. Exploratory Application and Relevance for Design and Integrity Assessments

To take advantage of unevenness, the component being designed must present regions loaded predominantly and permanently under compression. An example used here as a case study is the mechanical joining system called snap-fit (Fig. 5a). It behaves as a cantilever beam and Fig. 5b presents a usual cross section, which is rectangular with neutral axis in the middle of its height (h). From Bernoulli’s and elasticity theories[20,21], bending (σ) and shear (τ) stresses are maximum in the ABC plane and calculated neglecting stress concentration factors as:

\[ \sigma = \frac{F \cdot L \cdot h}{2 \cdot I}, \quad \tau = \frac{F \cdot Q}{I \cdot t} \]  

where force \( F = 66 \, N \), length \( L = 25 \, mm \), \( I \) represents the moment of inertia, \( Q \) represents the static moment of area and \( t \) represents the width in the analyzed vertical position[21].

Based on Eq. (8), equivalent stresses can be computed using Eqs. (3), (5) and (6). Consequently, safety factors (S.F.) can be computed as \( S.F. = \sigma_{eq} / \sigma_{equivalent} \), where \( \sigma_{equivalent} \) can be \( \sigma_{\text{abs}}, \sigma_{\text{abs,c}}, \text{ or } \sigma_{\text{abs,P}} \).

Maximum bending stresses occur at the top and bottom fibers of the cross section, while maximum shear stress occurs at the neutral axis. Consequently, these three positions are analyzed here and characterize structural integrity. Due to technological interest, PP was selected for the study. Considering true stress-strain data and 2% offset (representative for PP - see Table 1), \( E = 1730 \, MPa \) (average between tension and compression moduli), \( \sigma_{\text{abs,2.0,p}} = 33.0 \, MPa \), \( \sigma_{\text{abs,2.0,c}} = 40.9 \, MPa \) and unevenness level is \( m = 1.24 \).

To take advantage of uneven properties, the original cross section was optimized (Figs. 5b to 5d) to enhance the percentage of compressed areas. The parabolically modified criterion was adopted (Eq. (6)) due to the aforementioned experimental better agreement according to the literature[17,17]. Table 2 contains all the achieved results and will be discussed next.

The original rectangular cross section (Fig. 5b, Sec. #1) presents maximum normal stresses at the top and bottom fibers with same values but opposite signs, and maximum shear stresses at the neutral axis. The second column of Table 2 shows that original von Mises criterion (\( vM \)) leads to the same equivalent stresses at the top (\( \sigma_{\text{eq,up}} \)) and bottom (\( \sigma_{\text{eq,bottom}} \)) and unitary safety factors (\( S.F._{\text{up}} = S.F._{\text{bottom}} = 1.00 \)).

When considering the parabolically modified criterion (\( vM-P \)), it can be realized, in the third column, that top fibers (under tension) are not affected by the modified criterion, maintaining \( S.F._{\text{up}} = 1.00 \). On the other hand, for the neutral axis (under shear) and bottom fibers (under compression) safety factors are respectively \( S.F._{\text{up}} = 12.86 \) and \( S.F._{\text{bottom}} = 1.24 \). Taking the bottom fiber as an example, this occurrence demonstrates that there is 24% extra safety that was not accounted for by original von Mises criterion and can be optimized. This is in perfect agreement with the gains predictions from Fig. 4b, where round and square markers denote stresses at the top (+\( \sigma_{\text{C}} \)) and bottom (-\( \sigma_{\text{C}} \)) fibers respectively.

As the bottom fibers of the snap-fit operate under compression and present higher yield strength, one simple idea to illustrate the methodology is to turn the rectangular cross section into a trapezoidal one (Fig. 5c, Sec. #2). This geometrical change offsets neutral axis to a higher position and makes normal stresses at the bottom (compressive) larger than at the top (tensile). To determine geometric features for the cross section, a reduced gradient nonlinear optimization code (GRG2) was developed[22]. The code enforced original stiffness (by keeping \( I = 125 \, mm^4 \)) and unitary safety factors at the top and bottom considering von Mises parabolic model, providing the other geometrical features to configure an optimum trapezoid. Results are presented in Table 2, columns 4 and 5. See that original von Mises model predicts failure at the bottom (\( S.F._{\text{bottom}} = 0.81 \)), while parabolic von Mises model predicts \( S.F._{\text{bottom}} = 1.00 \). Consequently, keeping original stiffness and desired unitary safety factors, a mass reduction (area reduction) of 16.70% was achieved.

However, the trapezoid was not interesting to improve the neutral axis region and safety factors were kept extremely high (larger than 10). For illustration purposes (neglecting cost or manufacturability at this moment), a third cross section (Fig. 5d, Sec. #3) was obtained using the same optimization techniques but allowing the algorithm to count on two trapezoids. A minimum neutral axis width was specified in 2 mm to avoid elastic instability and the other geometrical features emerged. The last two columns of Table 2 show that keeping the same original stiffness and safety factors, the mass reduction in this case achieved 39.8%.

6. Concluding Remarks

From this work it is possible to conclude that:
- Conically modified theory is more sensitive to unevenness. However, for values up to \( m = 1.30 \) predictions from conical and parabolic models are essentially similar;

![Fig. 5](image-url)


Table 2 Results for the evaluated cross sections presented by Fig. 5. In each case, stresses and safety factors were computed using conventional (vM) and parabolically modified (vM-P) von Mises

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Section # 1</th>
<th>Section # 2</th>
<th>Section # 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{eq-up}$ (MPa)</td>
<td>33.00</td>
<td>33.00</td>
<td>33.00</td>
</tr>
<tr>
<td>$\sigma_{eq-axis}$ (MPa)</td>
<td>2.86</td>
<td>2.57</td>
<td>3.34</td>
</tr>
<tr>
<td>$\sigma_{eq-bottom}$ (MPa)</td>
<td>33.00</td>
<td>26.63</td>
<td>40.90</td>
</tr>
<tr>
<td>$S.F._{up}$</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$S.F._{axis}$</td>
<td>11.55</td>
<td>12.86</td>
<td>9.89</td>
</tr>
<tr>
<td>$S.F._{bottom}$</td>
<td>1.00</td>
<td>1.24</td>
<td>0.81</td>
</tr>
<tr>
<td>$I$ (mm$^4$)</td>
<td>125 (reference)</td>
<td>125 (+ 0.00%)</td>
<td>125 (+ 0.00%)</td>
</tr>
<tr>
<td>Section Area (mm$^2$)</td>
<td>60.00 (reference)</td>
<td>50.00 (~ 16.70%)</td>
<td>36.10 (~ 39.80%)</td>
</tr>
</tbody>
</table>

- considering deviation, only PA-66 presented even yield strength. PA-6, PP and HDPE presented unevenness levels between 21% and 36% considering engineering and 10% and 24% considering true properties. Considering true stress-strain data is recommended for realism and safety in integrity assessments;
- all tested materials presented stiffness reduction under compression, which must be considered for design and serve future phenomenological investigation;
- the incorporation of uneven polymer mechanical properties in modified von Mises criteria provided mass reductions up to 39.8% keeping original stiffness and safety factors, which proves that shape optimization based on pressure dependent criteria and unevenness levels can provide mass reductions higher than the simple difference between tensile and compressive properties itself. It encourages future developments in the field including additional practical testing and validation.

Acknowledgments

This investigation is supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) through funding 11/07121-2 and by Fundação Educacional Inaciana (FEI, Brazil) through materials and human resources.

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